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Fuel-Optimal Rendezvous in a General Central Force Field

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Introduction

IN a sequence of recent papers,[1-3] the fuel-optimal rendezvous problem in the Newtonian gravitational field of a central body has been investigated in detail. The results obtained generalize previous ones.[4-10] To derive these results the equations describing the system were linearized and then reduced effectively to one second-order equation. This reduction coupled with the application of the Pontryagin maximum principle then yielded the optimal rendezvous trajectory.

The objective of this Note is to show that these results can be generalized to any central force field. In particular, we demonstrate that the effective reduction of the linearized equations of motion to one second-order equation is true for any central force field. Using this fact, we generalize the results for the optimal rendezvous trajectory that were obtained in Refs. 1-3.

The plan of the presentation is as follows: first, we derive the equations of motion of the system and then we treat the rendezvous problem with and without the assumption of constant spacecraft mass.

Equations of Motion

The system we wish to consider consists of a satellite in orbit and a spacecraft in a central force field. The position of the satellite with respect to the field source is given by $\boldsymbol{R}(t)$ and its equation of motion is

$$\ddot{\boldsymbol{R}}(t) = -f(R)\boldsymbol{R}, \qquad R = (\boldsymbol{R}\cdot\boldsymbol{R})^{1/2} \tag{1}$$

The position of the spacecraft measured from the satellite is $\boldsymbol{r}(t)$ and its equation of motion is

$$\begin{aligned}\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} &= -f\Big\{\big[(\boldsymbol{R}+\boldsymbol{r})\cdot(\boldsymbol{R}+\boldsymbol{r})\big]^{1/2}\Big\}(\boldsymbol{R}+\boldsymbol{r}) + \boldsymbol{T}/m \\ &= -f\left\{R\left(1+\frac{2\boldsymbol{R}\cdot\boldsymbol{r}}{R^2}+\frac{r^2}{R^2}\right)^{1/2}\right\}(\boldsymbol{R}+\boldsymbol{r}) + \boldsymbol{T}/m\end{aligned} \tag{2}$$

where $\boldsymbol{T}$ is the thrust.

To linearize the equations of motion of the spacecraft for $|r/R| \ll 1$, we make a first-order Taylor expansion of f

$$f\Big\{\big[(\boldsymbol{R}+\boldsymbol{r})\cdot(\boldsymbol{R}+\boldsymbol{r})\big]^{1/2}\Big\} = f(R) + f'(R)\left(\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R} + \frac{r^2}{2R}\right) + \cdots \tag{3}$$

Substituting Eq. (3) into Eq. (2) using Eq. (1) and linearizing, we obtain

$$\ddot{\boldsymbol{r}} = -f(R)\boldsymbol{r} - f'(R)\big[(\boldsymbol{R}\cdot\boldsymbol{r})/R\big]\boldsymbol{R} + \boldsymbol{T}/m \tag{4}$$

We now transform Eq. (4) to a rotating frame fixed in the satellite. This yields

$$\begin{aligned}\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r} &= -f(R)\boldsymbol{r} \\ &\quad - f'(R)\big[(\boldsymbol{R}\cdot\boldsymbol{r})/R\big]\boldsymbol{R} + \boldsymbol{T}/m\end{aligned} \tag{5}$$

where $\boldsymbol{\Omega}$ is the orbital angular velocity of the satellite.

However as the motion of the satellite is planar, we can choose the coordinate system attached to it so that the x_1 axis is tangential but opposed to the motion of the satellite, the x_2 axis is in the direction of $\boldsymbol{R}$, and x_3 completes a right-handed system. In this frame, we have $\boldsymbol{r} = (x_1, x_2, x_3)$, $\boldsymbol{R} = R(0,1,0)$, and $\boldsymbol{\Omega} = (0,0,\dot{\theta})$ (where θ is the true anomaly of the satellite). Using this setting and the law of angular momentum conservation

$$R^2\dot{\theta} = R^2\omega = \text{const} = L \tag{6}$$

Equation (5) becomes

$$\ddot{x}_1 = \big[\omega^2 - h(\omega)\big]x + \dot{\omega}x_2 + 2\omega\dot{x}_2 + T_1/m \tag{7a}$$

$$\ddot{x}_2 = -\dot{\omega}x_1 + \big[\omega^2 - h(\omega) - g(\omega)\big]x_2 - 2\omega\dot{x}_1 + T_2/m \tag{7b}$$

$$\ddot{x}_3 = -h(\omega)x_3 + T_3/m \tag{7c}$$

where

$$h(\omega) = f(R), \qquad g(\omega) = f'(R)R \tag{8}$$

Changing variables from t to θ, we obtain

$$\omega^2 x_1'' + \omega\omega' x_1' = \big[\omega^2 - h(\omega)\big]x_1 + \omega\omega' x_2 + 2\omega^2 x_2' + T_1/m \tag{9a}$$

$$\begin{aligned}\omega^2 x_2'' + \omega\omega' x_2' &= -\omega\omega' x_1 + \big[\omega^2 - h(\omega) - g(\omega)\big]x_2 \\ &\quad - 2\omega^2 x_1' + T_2/m\end{aligned} \tag{9b}$$

$$\omega^2 x_3'' + \omega\omega' x_3 = -h(\omega)x_3 + T_3/m \tag{9c}$$

where primes denote differentiation with respect to θ.

To simplify these equations, we now make the transformation

$$y_i = \omega^{1/2}x_i, \qquad i = 1, 2, 3 \tag{10}$$

This leads to

$$y_1'' - M(\omega)y_1 = 2y_2' + T_1/(m\omega^{3/2}) \tag{11a}$$

$$y_2'' - M(\omega)y_2 - G(\omega)y_2 = -2y_1' + T_2/(m\omega^{3/2}) \tag{11b}$$

$$y_3'' - I(\omega)y_3 = T_3/(m\omega^{3/2}) \tag{11c}$$

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where

$$M(\omega) = \omega^{-3/2}\left\{\left[\omega^2 - h(\omega)\right]\omega^{-1/2} + \omega/2(\omega'\omega^{-1/2})'\right\} \tag{12a}$$

$$G(\omega) = -\omega^{-2}g(\omega) \tag{12b}$$

$$I(\omega) = \omega^{-1/2}\left\{-\omega^{-3/2}h(\omega) + 1/2(\omega'\omega^{-1/2})'\right\} \tag{12c}$$

We now prove that $M(\omega)=0$ and $I(\omega)=-1$. In fact, using Eq. (6) we have

$$\omega^{1/2} = L^{1/2}u, \qquad u = 1/R \tag{13}$$

Applying Eqs. (13) and (8) to $M(\omega)$, we have

$$\begin{aligned}&\left\{\left[\omega^2 - h(\omega)\right]\omega^{-1/2} + \omega/2(\omega'\omega^{-1/2})'\right\}\\ &= \omega\left[(\omega^{1/2})'' + \omega^{1/2} - \omega^{-3/2}h(\omega)\right]\\ &= L^{-1/2}\left\{L^2u^2\left[u'' + u\right] - \left[f(1/u)/u\right]\right\}\end{aligned} \tag{14}$$

The last expression in Eq. (14) is the standard equation of motion for a particle in orbit in a central force field[11] and hence is zero. The proof that $I(\omega)=-1$ is similar.

Thus, we have shown that the linearized equations of motion take the form

$$y_1'' = 2y_2' + T_1/m\omega^{3/2} \tag{15a}$$

$$y_2'' - G(\omega)y_2 = -2y_1' + T_2/m\omega^{3/2} \tag{15b}$$

$$y_3'' + y_3 = T_3/m\omega^{3/2} \tag{15c}$$

We observe that when $\boldsymbol{T}=0$, Eq. (15) reduces essentially to one nontrivial equation

$$y_2'' - G(\omega)y_2 + 4y_2 = \alpha \tag{16}$$

where α is an arbitrary constant.

Control Problem for the Linearized Equations

As Eq. (15) closely resembles the equations derived in Refs. 1 and 3 for Keplerian orbits, the discussion and results related to the control problem closely follow those that appeared in these publications. Accordingly, we make only a concise presentation of the results.

Constant Mass Problem

In this case, we assume that the mass loss to the spacecraft during the rendezvous maneuvers can be ignored. The fuel-optimal rendezvous problem consists then of minimizing

$$J(\boldsymbol{T}) = \int_{\theta_0}^{\theta_f} |\boldsymbol{T}(\theta)|\,\mathrm{d}\theta/\omega \tag{17}$$

with the boundary conditions

$$y(\theta_0) = y_0, \qquad y'(\theta_0) = v_0 \tag{18a}$$

$$y(\theta_f) = y_f, \qquad y'(\theta_f) = v_f \tag{18b}$$

subject to the equations

$$y'(\theta) = v(\theta) \tag{19a}$$

$$v'(\theta) = A(\theta)y(\theta) + B(\theta)v(\theta) + b\omega^{-3/2}\boldsymbol{u}(\theta) \tag{19b}$$

where $u(\theta)$ is the normalized thrust $\boldsymbol{u}(\theta) = \boldsymbol{T}(\theta)/T_{\max}$ ($|\boldsymbol{u}(\theta)| \le 1$) and

$$A = \begin{bmatrix} 0 & 0 & 0\\ 0 & G & 0\\ 0 & 0 & -1\end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 0\\ -2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{20}$$

Since the equations of motion for this system are linear, the Pontryagin principle yields necessary and sufficient conditions for optimality. The Hamiltonian of the system is

$$\begin{aligned}H(\boldsymbol{u}_0) = \frac{\ell|\boldsymbol{u}_0|}{\omega} + \boldsymbol{p}(\theta)^T\cdot v(\theta) + \boldsymbol{q}(\theta)^T\\ \times\left[A(\theta)y(\theta) + Bv(\theta) + b\boldsymbol{u}_0/\omega^{1/2}\right]\end{aligned} \tag{21}$$

where $\ell>0$ is a parameter and $\boldsymbol{p}$ and $\boldsymbol{q}$ (the conjugate variable) satisfy

$$\boldsymbol{p}' = \partial H/\partial y = -A^T(\theta)\boldsymbol{q}(\theta) \tag{22}$$

$$\boldsymbol{q}' = -\partial H/\partial v = -\boldsymbol{p}(\theta) - B^T\boldsymbol{q}(\theta) \tag{23}$$

The optimal control $\boldsymbol{u}(\theta)$ must then minimize the expression

$$b/\omega\left(|\boldsymbol{u}_0| + (\boldsymbol{q}^T\cdot\boldsymbol{u}_0)/\omega^{1/2}\right) \tag{24}$$

Hence we infer that

$$\boldsymbol{u}(\theta) = \begin{cases} 0 & |\boldsymbol{q}(\theta)|/\omega^{1/2} < 1\\ -\boldsymbol{q}(\theta)/|\boldsymbol{q}(\theta)| & |\boldsymbol{q}(\theta)|/\omega^{1/2} > 1\\ -\boldsymbol{q}(\theta)f(\theta)/\omega^{1/2} & |\boldsymbol{q}(\theta)|/\omega^{1/2} = 1\end{cases} \tag{25}$$

where $0\le f(\theta)\le 1$ and the primer vector $\boldsymbol{q}$ satisfies the differential equation

$$\boldsymbol{q}'' - B\boldsymbol{q}' - A\boldsymbol{q} = 0 \tag{26}$$

which is the same equation as Eq. (19) with $\boldsymbol{u}=0$ and hence equivalent to Eq. (16).

Varying Mass Problem

When the mass loss of the pursuing spacecraft must be taken into account, Eqs. (17), (18), and (19) must be supplemented by

$$m'(\theta) = k|\boldsymbol{u}(\theta)|/\omega \tag{27}$$

The resulting control problem is not linear and the Pontryagin principle yields only extreme solutions to the control problem. The Hamiltonian is given by

$$\begin{aligned}H(\boldsymbol{u}_0) = \ell_0\boldsymbol{u}_0/\omega + \boldsymbol{p}^T\cdot v + \boldsymbol{q}^T\\ \times\left[Ay + Bv + b\boldsymbol{u}_0(\theta)/\omega^{3/2}m(\theta)\right] - r(\theta)k|\boldsymbol{u}_0|/\omega\end{aligned} \tag{28}$$

where $|\boldsymbol{u}_0|\le 1$ and the conjugate variables $\boldsymbol{p}$, $\boldsymbol{q}$, and r must satisfy Eqs. (22) and (23) and

$$r'(\theta) = [b\boldsymbol{q}^T\cdot\boldsymbol{u}(\theta)]/\omega^{3/2}m^2(\theta) \tag{29}$$

H is minimum when $\boldsymbol{u}(\theta)$ minimizes the expression

$$\left[\ell_0 - kr(\theta)\right]|\boldsymbol{u}_0| + (b\boldsymbol{q}^T\cdot\boldsymbol{u}_0)/\omega^{1/2}m(\theta) \tag{30}$$

For $\boldsymbol{q}\ne 0$, this happens when

$$\boldsymbol{u}_0 = \ell_u(\theta)f(\theta) \tag{31}$$

where $0\le f(\theta)\le 1$ and $\ell_u = -\boldsymbol{q}/|\boldsymbol{q}|$.

In summary, we observe that in the gravitational case, Eq. (19) [or equivalently Eq. (6)] can be solved in closed form for all Keplerian orbits. As a result, the optimal rendezvous can be computed in closed form. For the general central force field this is obviously impossible. In this case it is necessary to compute $\boldsymbol{q}$ numerically. However, the effective reduction of this problem to one differential equation is significant from both a theoretical and a practical point of view.

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Fuel Optimal Reorientation of Axisymmetric Spin-Stabilized Satellites

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I. Introduction

THE problem of fuel optimal reorientation of axisymmetric spin-stabilized rigid satellites using reaction control jets was widely investigated in the 1960s. Adopting various approaches, researchers concluded that fuel optimal reorientation requires two impulsive torques when the maneuver time is large enough. However, such two-impulse schemes become invalid for small maneuver times. This Note applies a new numerical technique to solve the fuel optimal spin axis reorientation problem for large and small maneuver times.

For simplicity, early investigations of reorientation maneuvers employed impulsive torques with no implication of fuel optimality.[1] Later, the two-impulse scheme was shown to be fuel optimal for maneuver times that are large enough to accommodate the required precessional motion.[2] Next, Yin and Grimmel considered the bounded input problem, showed that the optimal control was bang off bang, and developed the associated switching curves. Furthermore, it was shown that reducing the maneuver time decreases the control off time until the fuel optimal control converges to the time optimal bang-bang control.[3] Later, the bounded fuel optimal problem was examined in detail by Wingate, who determined reorientation schemes for various geometries by numerically minimizing the associated Hamiltonian.[4] Recently, a new technique was developed for exactly solving unbounded fuel optimal control problems for arbitrary maneuver times.

Extending the work of Krasovskii,[5] Neustadt developed a geometric approach to solving optimal control problems.[6] This approach was applied to fuel optimal control and resulted in a control scheme that was demonstrated on second-order systems.[7] Recently, the development of an adaptive grid bisection search has made it possible to exactly solve fuel optimal control problems for higher order systems with a prescribed maneuver time.[8] This method is well suited for the rigid satellite problem, especially when rapid reorientation is desired.

The formulation of the fuel optimal reorientation of a spin-stabilized axisymmetric rigid satellite is presented in Sec. II. Nondimensional plots of fuel consumption vs maneuver time for various geometries are given in Sec. III, and some typical maneuvers are detailed. Finally, some concluding remarks are given in Sec. IV.

II. Fuel Optimal Reorientation

A rigid satellite is shown in Fig. 1. The inertial coordinates i_1, i_2, and i_3 and the body-fixed coordinates b_1, b_2, and b_3 are related by successive rotations through an appropriate selection of Euler angles. The coordinate system is first rotated about the i_3 axis through the angle ψ. Next, the system is rotated about the nonspinning body y axis through the angle θ. Finally, the system is rotated about the body axis of symmetry b_1 thorugh the angle ϕ. The satellite is spinning at the rate Ω about the axis of symmetry b_1 that has an associated mass moment of inertia I_a. The transverse mass moment of inertia is denoted by I_t. The satellite is reoriented by means of body-fixed reaction control jets producing bidirectional control moments about the b_3 axis. Assuming that the control moment is initially aligned with the nonspinning body z axis, the linearized equations of motion are[4]

$$\frac{d^2\psi(\tau)}{d\tau^2} - r\frac{d\theta(\tau)}{d\tau} = \cos\tau u(\tau)$$

$$\frac{d^2\theta(\tau)}{d\tau^2} + r\frac{d\psi(\tau)}{d\tau} = \sin\tau u(\tau) \tag{1}$$

where $\tau = \Omega t$ is the nondimensional time, $u(\tau) = M(\tau)/I_t\Omega^2$ is the admissible nondimensional control moment, and $r = I_a/I_t$ is the inertia ratio. The equation of motion was linearized assuming that θ and $d\psi/d\tau$ are small. Note that r ranges from 0 (long thin rod) to 2 (flat disk).

We define the state vector $x(\tau) = [\psi(\tau)\ \dot\psi(\tau)\ \theta(\tau)\ \dot\theta(\tau)]^T$ and convert Eq. (1) to the first-order form

$$\dot{x}(\tau) = Ax(\tau) + b(\tau)u(\tau) \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \\ 0 & -r & 0 & 0 \end{bmatrix} \tag{3a}$$

$$b(\tau) = \begin{bmatrix} 0 \\ \cos\tau \\ 0 \\ -\sin\tau \end{bmatrix} \tag{3b}$$

The objective of the control is to transfer the system from some initial state x_0 to a final state x_1 in maneuver time τ_f while minimizing the fuel

$$\text{fuel} = \int_0^{\tau_f} |u(\tau)|\, d\tau \tag{4}$$

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